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Evolution of transverse momentum dependent distribution and fragmentation functions

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Abstract

We use Lorentz invariance and the QCD equations of motion to study the evolution of functions that appear at leading order in a $1/Q$ expansion in azimuthal asymmetries. This includes the evolution equation of the Collins fragmentation function. The moments of these functions are matrix elements of known twist two and twist three operators. We present the evolution in the large N_c limit, restricting to non-singlet for the chiral-even functions. © 2002 Published by Elsevier Science B.V.

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1. Introduction

Azimuthal asymmetries in hard scattering processes with at least two relevant hadrons constitute a rich phenomenology, suitable for studying quark and gluon correlations in hadrons. By relevant hadrons we mean hadrons used as target or detected in the final state. A well-known azimuthal asymmetry appears in the semi-inclusive deep inelastic polarized leptoproduction of pions ($ep^{\uparrow} \to e'\pi X$) generated by the so-called Collins effect [1]. This asymmetry is one of the few possibilities to gain access to the so-called transversity or transverse spin distribution function [2,3], which is the third distribution function needed for the complete characterization of the (collinear) spin state of a proton as probed in hard scattering processes. In contrast to the transversity function, the evolution of the Collins fragmentation function has not been presented yet. Knowledge of this evolution is indispensable for relating measurements at different energies.

In processes like the semi-inclusive leptoproduction mentioned above, it is important to take transverse momentum of partons into account. The parton distribution functions

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as a function of a light-cone momentum fraction and transverse momentum have first been studied by Ralston and Soper [2] for the Drell–Yan process at tree level. The precise operator definition of such transverse momentum (p_T) dependent distribution (and also fragmentation) functions is a non-trivial issue (mainly because of gauge invariance) and in several studies this matter has been addressed [4–7]. Besides their definition and appearance in cross sections, they have not yet been considered beyond tree level. One thing that one wants to know is how these p_T dependent distribution and fragmentation functions evolve, for instance the fragmentation function pointed out by Collins, which is one of the new functions entering in the description of hadrons when transverse momentum is considered. Its evolution will be one of the new results presented in this paper, although we limit ourselves to the large N_c limit, in which case the evolution is an autonomous one.

In general, factorization of hard scattering processes means that it is possible to separate parts containing only soft or hard physics. In the perturbative calculation of the hard subprocess (the partonic cross section) one encounters collinear divergences, which one can absorb into the matrix elements (or equivalently the distribution or fragmentation functions) describing the soft part of the process. This redefinition of the soft parts should be possible to all orders in the coupling constant. This procedure introduces a factorization scale and the goal is to calculate the dependence on this scale, which determines the high-energy behavior of the cross section.

Factorization crucially depends on the presence of a large energy scale in the process, such as the space-like momentum transfer squared in leptoproduction or the time-like momentum squared of a lepton pair in Drell–Yan scattering. In this paper we will be concerned with functions that appear in processes which have, apart from such a hard scale, an additional soft momentum scale, related to the transverse momentum of the partons. The first factorization theorem for such a situation was obtained in Ref. [4] for the process $e^+e^- \to h_1 h_2 X$, where the vector boson has a large invariant mass Q, but a small transverse momentum Q_T with respect to the two almost back-to-back hadrons h_1 and h_2, i.e., $Q_T^2 \ll Q^2$. Similar situations occur in the Drell–Yan process, where one has besides the momentum of the lepton pair, two hadron momenta and in one-hadron inclusive leptoproduction where one also deals with three momenta: the large momentum transfer, the target momentum and the momentum of the produced hadron.

The effects of parton transverse momenta lead to azimuthal asymmetries in such processes, often coupled to the spin of the partons and/or hadrons. Just as for spin asymmetries, the azimuthal asymmetries provide a rich new phenomenology in Drell–Yan scattering, semi-inclusive deep inelastic scattering and electron–positron annihilation [2, 8–13]. In this paper we will study the scale dependence of the various distribution and fragmentation functions appearing in these (polarized) processes. We do this for specific moments in both p_T and x, employing Lorentz invariance and the QCD equations of motion. The moments in x for leading (collinear) distribution functions (appearing for instance in inclusive leptoproduction) are related to matrix elements of twist two operators. On the other hand, for the transverse moments entering the azimuthal asymmetry expressions of interest, one finds relations to matrix elements of twist two *and* twist three operators, for which the evolution, however, is known. In the large N_c limit this evolution becomes particularly simple and is known from studies of inclusive processes where the transverse moments can be eliminated from the expressions. This knowledge allows us to obtain the evolution equations for the desired p_T and x moments of the transverse momentum dependent distribution and fragmentation functions, that enter in azimuthal (spin) asymmetries. Such asymmetries recently have gained in interest, as can be seen from the experimental studies in Refs. [14–16].

2. Formalism

In this paper we will study the scale dependence of the distribution and fragmentation functions in (for example) one-hadron inclusive leptoproduction ($eH \to ehX$) at leading order in an expansion in $1/Q$, where $q^2 = -Q^2$ is the space-like momentum transfer squared. Experimentally, we are interested in azimuthal asymmetries in the current fragmentation region, in which case the target hadron momentum P and the produced hadron momentum P_h satisfy $P \cdot P_h \sim Q^2$. We introduce two light-like vectors via the hadron momenta and parameterize

$$P = \frac{\xi M^2}{\widetilde{Q}\sqrt{2}} n_- + \frac{\widetilde{Q}}{\xi\sqrt{2}} n_+, \tag{1}$$

$$P_h = \frac{\zeta \widetilde{Q}}{\sqrt{2}} n_- + \frac{M_h^2}{\zeta \widetilde{Q}\sqrt{2}} n_+, \tag{2}$$

$$q = \frac{\widetilde{Q}}{\sqrt{2}} n_- - \frac{\widetilde{Q}}{\sqrt{2}} n_+ + q_T, \tag{3}$$

where n_+ and n_- are two light-like vectors, chosen such that $n_+ \cdot n_- = 1$. We will often refer to the $\pm$ components of a vector p, which are defined as $p^{\pm} = p \cdot n_{\mp}$. We define the transverse momentum scale $Q_T^2 = -q_T^2$. We are interested in the region where $Q_T^2 \ll Q^2$. Up to mass corrections of order $1/Q^2$ one then has $\widetilde{Q}^2 = Q^2 - Q_T^2 \approx Q^2$. The ratio $\xi = -q^+/P^+ \approx Q^2/2P \cdot q = x_{\mathrm{B}}$ is the Bjorken scaling variable and the ratio $\zeta = P_h^-/q^- \approx P \cdot P_h/P \cdot q = z_h$ is the usual fragmentation variable.

In the case of inclusive deep inelastic scattering the soft part of the process is described by a correlation function. To be more specific, in leading order in powers of $1/Q$, the forward scattering Compton amplitude $T^{\mu\nu}$ can be written as

$$T^{\mu\nu} = \int dx\, \mathrm{Tr}\big[S^{\mu\nu}(x)\Phi(x)\big] + \cdots. \tag{4}$$

Here, $S^{\mu\nu}$ and Φ are the hard and soft scattering parts, respectively. The hard part $S^{\mu\nu}$ is the γ^*-quark forward scattering amplitude. The color gauge invariant soft part is defined as

$$\Phi_{ij}(x) \equiv \int \frac{d\xi^-}{2\pi} e^{ip\cdot\xi} \langle P, S|\bar{\psi}_j(0)\,\mathcal{U}(0,\xi)\psi_i(\xi)|P, S\rangle\Big|_{\mathrm{LC}}, \tag{5}$$

where the subscript ‘LC’ indicates $\xi^+ = \xi_T = 0$ and

$$\mathcal{U}(0,\xi) = \mathcal{P}\exp\left(-ig\int_0^{\xi^-} d\eta^-\, A^+(\eta)\right), \tag{6}$$

is a gauge link with the path running along n_-. The correlation function Φ is a function of the light-cone momentum fraction $x = p^+/P^+$ only. The most general parameterization of Φ which is in accordance with the required symmetries (hermiticity, parity, time reversal) and which is relevant for calculations at leading order in $1/Q$ is given by

$$\Phi^{\text{twist-2}}(x) = \frac{1}{2}\left\{f_1(x)\not{n}_+ + S_L g_1(x)\gamma_5\not{n}_+ + h_1(x)\gamma_5\not{S}_T\not{n}_+\right\}, \tag{7}$$

where also for the spin vector a decomposition in $n_\pm$ is adopted, $S = S_L(P^+/M)n_+ - S_L(P^-/M)n_- + S_T$. Specifying also the flavor one also encounters the notations $q(x) = f_1^q(x)$, $\Delta q(x) = g_1^q(x)$ and $\delta q(x) = \Delta_T q(x) = h_1^q(x)$. The evolution equations for these functions are known to next-to-leading order [17,18] and for the singlet f_1 and g_1 there is mixing with the unpolarized and polarized gluon distribution functions $g(x)$ and $\Delta g(x)$, respectively.

Denoting these functions as twist-2 makes sense because the local operators connected to the Mellin moments of these functions are related to the matrix elements of local twist-2 operators, like $\bar{\psi}\gamma^+(D^+)^n\psi$.

If one calculates $T_{\mu\nu}$ up to order $1/Q$ one needs also the M/P^+ parts in the parameterization of $\Phi(x)$ [3,11]

$$\begin{aligned}\Phi^{\text{twist-3}}(x) = {}& \frac{M}{2P^+}\left\{e(x) + g_T(x)\gamma_5\not{S}_T + S_L h_L(x)\gamma_5\frac{[\not{n}_+,\not{n}_-]}{2}\right\}\\ &+ \frac{M}{2P^+}\left\{-iS_L e_L(x)\gamma_5 - f_T(x)\epsilon_T^{\rho\sigma}\gamma_\rho S_{T\sigma} + ih(x)\frac{[\not{n}_+,\not{n}_-]}{2}\right\}.\end{aligned} \tag{8}$$

We have not imposed time-reversal invariance in order to study also the T-odd functions, which are important, e.g., in the study of fragmentation functions. The functions e, g_T and h_L are T-even, the functions e_L, f_T and h are T-odd (we will not concern ourselves with the formal problems related to T-odd distribution functions [1,11]). The leading order evolution of e, g_T and h_L is known [19–22] and for the non-singlet case this also provides the evolution of the T-odd functions e_L, f_T and h, respectively, for which the operators involved differ only from those of the T-even functions by a γ_5 matrix.

The twist assignments of these functions is better seen by considering the light-cone correlators

$$\begin{aligned}\Phi^\alpha_{Dij}(x,y) \equiv {}& \frac{P^+}{M}\int\frac{d\xi^-}{2\pi}\frac{d\eta^-}{2\pi}\,e^{ip_1\cdot(\xi-\eta)+ip\cdot\eta}\\ &\times \langle P,S|\bar{\psi}_j(0)\mathcal{U}(0,\eta)iD_T^\alpha(\eta)\mathcal{U}(\eta,\xi)\psi_i(\xi)|P,S\rangle\Big|_{\text{LC}},\end{aligned} \tag{9}$$

depending on $x = p^+/P^+$ and $y = p_1^+/P^+$. It is parameterized in terms of two-argument functions [3]

$$\Phi_D^\alpha(x,y) = \frac{1}{2}\big\{G_D(x,y)i\epsilon_T^{\alpha\beta}S_{T\beta}\not{n}_+ + \widetilde{G}_D(x,y)S_T^\alpha\gamma_5\not{n}_+ + H_D(x,y)S_L\gamma_5\gamma_T^\alpha\not{n}_+ + E_D(x,y)\gamma_T^\alpha\not{n}_+\big\}, \tag{10}$$

where parity invariance has been imposed. Hermiticity leads to $G_D^*(x,y) = -G_D(y,x)$, $\widetilde{G}_D^*(x,y) = \widetilde{G}_D(y,x)$, $H_D^*(x,y) = H_D(y,x)$, and $E_D^*(x,y) = -E_D(y,x)$. Time reversal invariance would require these functions to be real. The QCD equations of motion can be used to relate the twist-3 functions appearing in the parameterization of Φ to the correlators Φ_D^α. This gives the relations

$$\int dy\,\big[G_D(x,y) + \widetilde{G}_D(x,y)\big] = xg_T(x) - \frac{m}{M}\,h_1(x) + ixf_T(x), \tag{11}$$

$$2\int dy\,H_D(x,y) = xh_L(x) - \frac{m}{M}\,g_1(x) - ixe_L(x), \tag{12}$$

$$2\int dy\,E_D(x,y) = xe(x) - \frac{m}{M}\,f_1(x) + ixh(x). \tag{13}$$

The local operator matrix elements corresponding with the moments of the functions in $\Phi_D^\alpha(x)$ are (note that α is transverse) twist-3 operators, up to quark mass contributions multiplying twist-2 operators.

3. Transverse momentum dependent distribution functions

If one considers a semi-inclusive hard scattering process in which two hadrons are identified (in either initial and/or final state), then the treatment of transverse momentum is important. One needs to study correlation functions that also depend on the transverse momentum, $\Phi(x, \boldsymbol{p}_T)$, for which the most general parameterization involves more functions. To be more specific, one needs the lightfront correlation function

$$\Phi_{ij}(x,\boldsymbol{p}_T) \equiv \int \frac{d\xi^-\,d^2\xi_T}{(2\pi)^3}\,e^{ip\cdot\xi}\,\langle P,S|\bar{\psi}_j(0)\,\mathcal{U}(0,-\infty)\,\mathcal{U}(-\infty,\xi)\,\psi_i(\xi)|P,S\rangle\bigg|_{\xi^+=0}. \tag{14}$$

At leading orders in powers of $1/Q$ the following transverse momentum dependent distribution functions are needed to parameterize this correlation function [2,11]

$$\Phi(x,\boldsymbol{p}_T) = \frac{1}{2}\Bigg\{f_1\big(x,\boldsymbol{p}_T^2\big)\not{n}_+ + f_{1T}^\perp\big(x,\boldsymbol{p}_T^2\big)\,\frac{\epsilon_{\mu\nu\rho\sigma}\gamma^\mu n_+^\nu p_T^\rho S_T^\sigma}{M} - g_{1s}(x,\boldsymbol{p}_T)\not{n}_+\gamma_5 - h_{1T}\big(x,\boldsymbol{p}_T^2\big)i\sigma_{\mu\nu}\gamma_5 S_T^\mu n_+^\nu - h_{1s}^\perp(x,\boldsymbol{p}_T)\,\frac{i\sigma_{\mu\nu}\gamma_5 p_T^\mu n_+^\nu}{M} + h_1^\perp\big(x,\boldsymbol{p}_T^2\big)\,\frac{\sigma_{\mu\nu}p_T^\mu n_+^\nu}{M}\Bigg\}. \tag{15}$$

We used the shorthand notation

$$g_{1s}(x,\boldsymbol{p}_T) \equiv S_L g_{1L}\big(x,\boldsymbol{p}_T^2\big) + \frac{(\boldsymbol{p_T}\cdot\boldsymbol{S}_T)}{M}\,g_{1T}\big(x,\boldsymbol{p}_T^2\big), \tag{16}$$

and similarly for $h_{1s}^\perp$. The parameterization contains two T-odd functions, the Sivers function $f_{1T}^\perp$ [23,24] and the function $h_1^\perp$, the distribution function analogue of the Collins fragmentation function [1]. Upon integration over $\boldsymbol{p_T}$, Eq. (15) reduces to $\Phi(x)$ with

$$f_1(x) \equiv \int d^2p_T\, f_1\big(x, \boldsymbol{p_T}^2\big), \tag{17}$$

$$g_1(x) \equiv \int d^2p_T\, g_{1L}\big(x, \boldsymbol{p_T}^2\big), \tag{18}$$

$$h_1(x) \equiv \int d^2p_T \left[h_{1T}\big(x, \boldsymbol{p_T}^2\big) + \frac{\boldsymbol{p_T}^2}{2M^2}\, h_{1T}^{\perp}\big(x, \boldsymbol{p_T}^2\big)\right]. \tag{19}$$

The deep inelastic scattering process is only sensitive to the latter three functions, but in semi-inclusive deep inelastic scattering or in the Drell–Yan process (at small q_T), one is sensitive to the p_T-dependent functions. At measured q_T one deals with a convolution of two p_T-dependent functions, where the transverse momenta of the partons from different hadrons combine to q_T [2,10,25]. A decoupling is achieved by studying cross-sections weighted with the momentum q_T^α, leaving only the directional (azimuthal) dependence. The functions that appear in that case are contained in

$$\Phi_\partial^\alpha(x) \equiv \int d^2p_T\, \frac{p_T^\alpha}{M}\, \Phi(x, \boldsymbol{p}_T) \tag{20}$$

which projects out the functions in $\Phi(x, \boldsymbol{p}_T)$ where p_T appears linearly,

$$\begin{aligned}\Phi_\partial^\alpha(x) = \frac{1}{2}\bigg\{ &-g_{1T}^{(1)}(x) S_T^\alpha \not{n}_+ \gamma_5 - S_L h_{1L}^{\perp(1)}(x)\, \frac{[\gamma^\alpha, \not{n}_+]\gamma_5}{2} \\ &- f_{1T}^{\perp(1)}(x) \epsilon^\alpha{}_{\mu\nu\rho}\gamma^\mu n_-^\nu S_T^\rho - i h_1^{\perp(1)}(x)\, \frac{[\gamma^\alpha, \not{n}_+]}{2}\bigg\},\end{aligned} \tag{21}$$

where we define $\boldsymbol{p}_T^2/2M^2$-moments (transverse moments) as

$$f^{(n)}(x) = \int d^2p_T \left(\frac{\boldsymbol{p}_T^2}{2M^2}\right)^n f(x, \boldsymbol{p}_T). \tag{22}$$

The functions $h_1^\perp$ and $f_{1T}^\perp$ are T-odd, vanishing if T-reversal invariance can be applied to the matrix element.

At this point one can invoke Lorentz invariance as a possibility to rewrite some functions. All functions in $\Phi(x)$ and $\Phi_\partial^\alpha(x)$ involve nonlocal matrix elements of two quark fields. Before constraining the matrix elements to the light-cone or lightfront only a limited number of amplitudes can be written down [10]. This leads to the following Lorentz-invariance relations [10,11,19]

$$g_T = g_1 + \frac{d}{dx}\, g_{1T}^{(1)}, \tag{23}$$

$$h_L = h_1 - \frac{d}{dx}\, h_{1L}^{\perp(1)}, \tag{24}$$

$$f_T = -\frac{d}{dx}\, f_{1T}^{\perp(1)}, \tag{25}$$

$$h = -\frac{d}{dx}\, h_1^{\perp(1)}. \tag{26}$$

From these relations, it is clear that the $\boldsymbol{p}_T^2/2M^2$ moments of the p_T-dependent functions, appearing in $\Phi_\partial^\alpha(x)$, involve both twist-2 and twist-3 operators.

Starting from the defining expression of Φ, one obtains, after weighting with p_T, the gauge invariant operator structure,

$$\begin{aligned} M\big(\Phi_\partial^\alpha\big)_{ij}(x) &= \int \frac{d\xi^-}{2\pi}\, e^{ip\cdot\xi} \langle P,S|i\partial_T^\alpha\big[\bar{\psi}_j(0)\mathcal{U}(0,-\infty)\mathcal{U}(-\infty,\xi)\psi_i(\xi)\big]|P,S\rangle\Big|_{\xi^+=\xi_T=0} \\ &= \int \frac{d\xi^-}{2\pi}\, e^{ip\cdot\xi} \Bigg\{ \langle P,S|\bar{\psi}_j(0)\mathcal{U}(0,\xi)iD_T^\alpha\psi_i(\xi)|P,S\rangle\Big|_{\xi^+=\xi_T=0} \\ &\qquad - \langle P,S|\bar{\psi}_j(0)\mathcal{U}(0,-\infty) \\ &\qquad \times \int_{-\infty}^{\xi^-} d\eta^-\, \mathcal{U}(-\infty,\eta)gF^{+\alpha}(\eta)\mathcal{U}(\eta,\xi)\psi_i(\xi)|P,S\rangle\Big|_{\xi^+=\xi_T=0}\Bigg\}. \end{aligned} \tag{27}$$

To see the partonic interpretation of the second term, consider the light-cone gauge ($A^+=0$), in which case $F^{+\alpha}=\partial_-A_T^\alpha$ and the gauge links become unity. Up to some for our purposes not relevant complications with boundary terms, the expression thus reduces to

$$\Phi_\partial^\alpha(x) = \int dy\,\big[\Phi_D^\alpha(x,y) - \Phi_A^\alpha(x,y)\big], \tag{28}$$

where the leading part of the Φ_A^α matrix element (in gauge $A^+=0$) is built from ψ_+ and A_T^α fields. The correlator Φ_A^α can be parameterized analogous to Φ_D^α with (interaction-dependent) functions G_A, $\widetilde{G}_A$, H_A and E_A with similar hermiticity properties as the functions in Φ_D^α. Using Eq. (28) we define the following combinations,

$$\begin{aligned} &\int dy\,\big[G_A(x,y)+\widetilde{G}_A(x,y)\big] \\ &\quad = xg_T(x) - \frac{m}{M}h_1(x) - g_{1T}^{(1)}(x) + i\big[xf_T(x)+f_{1T}^{\perp(1)}(x)\big] \equiv x\tilde{g}_T(x) + ix\tilde{f}_T(x), \end{aligned} \tag{29}$$

$$2\int dy\, H_A(x,y) = xh_L(x) - \frac{m}{M}g_1(x) + 2h_{1L}^{\perp(1)}(x) - ixe_L(x) \equiv x\tilde{h}_L(x) - ix\tilde{e}_L(x), \tag{30}$$

$$2\int dy\, E_A(x,y) = xe(x) - \frac{m}{M}f_1(x) + i\big[xh(x)+2h_1^{\perp(1)}(x)\big] \equiv x\tilde{e}(x) + ix\tilde{h}(x). \tag{31}$$

In principle, one can connect the functions defined here to those appearing in the treatments of Ellis, Furmanski and Petronzio [26] or to those in the treatment of Jaffe and Soldate [27]. We end this section by giving the relation to the functions used in a more recent treatment by Belitsky [28,29]; comparison of the equations of motion Eqs. (29)–(31) and Lorentz invariance relations Eqs. (23)–(26) with those given in Ref. [29] leads us to identify

$$g_{1T}^{(1)}(x) = \overline{K}(x) - \int_x^1 dy\, f(y), \tag{32}$$

$$x\tilde{g}_T(x) = \int dx'\, \overline{D}(x,x') + \int_x^1 dy\, f(y), \tag{33}$$

with $\overline{K}$, $\overline{D}$ as defined in Ref. [29] and

$$f(y) = \int dx'\, \frac{\overline{D}(y,x') + \overline{D}(x',y)}{x'-y}. \tag{34}$$

4. Relations between twist-3 functions and transverse moments

Using the equations of motion relations in Eqs. (29)–(31) and the relations based on Lorentz invariance in Eqs. (23)–(26), it is straightforward to relate the various twist-3 functions and the $\boldsymbol{p}_T^2/2M^2$ (transverse) moments of p_T-dependent functions. The results, grouping relevant combinations, are

$$g_T(x) = \int_x^1 dy\, \frac{g_1(y)}{y} + \frac{m}{M}\left[\frac{h_1(x)}{x} - \int_x^1 dy\, \frac{h_1(y)}{y^2}\right] + \left[\tilde{g}_T(x) - \int_x^1 dy\, \frac{\tilde{g}_T(y)}{y}\right], \tag{35}$$

$$\frac{g_{1T}^{(1)}(x)}{x} = \int_x^1 dy\, \frac{g_1(y)}{y} - \frac{m}{M}\int_x^1 dy\, \frac{h_1(y)}{y^2} - \int_x^1 dy\, \frac{\tilde{g}_T(y)}{y}, \tag{36}$$

$$\begin{aligned} h_L(x) &= 2x\int_x^1 dy\, \frac{h_1(y)}{y^2} + \frac{m}{M}\left[\frac{g_1(x)}{x} - 2x\int_x^1 dy\, \frac{g_1(y)}{y^3}\right] \\ &\quad + \left[\tilde{h}_L(x) - 2x\int_x^1 dy\, \frac{\tilde{h}_L(y)}{y^2}\right], \end{aligned} \tag{37}$$

$$\frac{h_{1L}^{\perp(1)}(x)}{x^2} = -\int_x^1 dy\, \frac{h_1(y)}{y^2} + \frac{m}{M}\int_x^1 dy\, \frac{g_1(y)}{y^3} + \int_x^1 dy\, \frac{\tilde{h}_L(y)}{y^2}, \tag{38}$$

$$e(x) = \tilde{e}(x) + \frac{m}{M}\,\frac{f_1(x)}{x}, \tag{39}$$

$$f_T(x) = \left[\tilde{f}_T(x) - \int_x^1 dy\, \frac{\tilde{f}_T(y)}{y}\right], \tag{40}$$

$$\frac{f_{1T}^{\perp(1)}(x)}{x} = \int_x^1 dy\, \frac{\tilde{f}_T(y)}{y}, \tag{41}$$

$$h(x) = \left[\tilde{h}(x) - 2x \int_x^1 dy\, \frac{\tilde{h}(y)}{y^2}\right], \tag{42}$$

$$\frac{h_1^{\perp(1)}(x)}{x^2} = \int_x^1 dy\, \frac{\tilde{h}(y)}{y^2}, \tag{43}$$

$$e_L(x) = \tilde{e}_L(x). \tag{44}$$

Note that often the combinations of tilde functions between brackets are denoted by a single 'interaction-dependent' function.

In order to study the evolution of these functions, we consider the moments $[f]_n \equiv \int dx\, x^{n-1} f(x)$, giving

$$[g_T]_n = \frac{1}{n}[g_1]_n + \frac{n-1}{n}[\tilde{g}_T]_n + \frac{m}{M}\frac{n-1}{n}[h_1]_{n-1}, \tag{45}$$

$$\left[g_{1T}^{(1)}\right]_n = \frac{1}{n+1}\left([g_1]_{n+1} - [\tilde{g}_T]_{n+1} - \frac{m}{M}[h_1]_n\right), \tag{46}$$

$$[h_L]_n = \frac{2}{n+1}[h_1]_n + \frac{n-1}{n+1}[\tilde{h}_L]_n + \frac{m}{M}\frac{n-1}{n+1}[g_1]_{n-1}, \tag{47}$$

$$\left[h_{1L}^{\perp(1)}\right]_n = -\frac{1}{n+2}\left([h_1]_{n+1} - [\tilde{h}_L]_{n+1} - \frac{m}{M}[g_1]_n\right), \tag{48}$$

$$[e]_n = [\tilde{e}]_n + \frac{m}{M}[f_1]_{n-1}, \tag{49}$$

$$[f_T]_n = \frac{n-1}{n}[\tilde{f}_T]_n, \tag{50}$$

$$\left[f_{1T}^{\perp(1)}\right]_n = \frac{1}{n+1}[\tilde{f}_T]_{n+1}, \tag{51}$$

$$[h]_n = \frac{n-1}{n+1}[\tilde{h}]_n, \tag{52}$$

$$\left[h_1^{\perp(1)}\right]_n = \frac{1}{n+2}[\tilde{h}]_{n+1}, \tag{53}$$

$$[e_L]_n = [\tilde{e}_L]_n. \tag{54}$$

Actually, we need not consider the five T-odd functions separately. They can be simply considered as imaginary parts of other functions, when we allow complex functions. In particular one can expand the correlation functions into matrices in Dirac space [30] to show that the relevant combinations are $(g_{1T} - i f_{1T}^{\perp})$ which we can treat together as one complex function g_{1T}. Similarly we can use complex functions $(h_{1L}^{\perp} + i h_1^{\perp}) \to h_{1L}^{\perp}$, $(g_T + i f_T) \to g_T$, $(h_L + i h) \to h_L$, $(e + i e_L) \to e$.

5. Evolution equations

In case of autonomous evolution of a function f one has

$$\frac{d}{d\tau} f(x,\tau) = \frac{\alpha_s(\tau)}{2\pi} \int_x^1 \frac{dy}{y} P^{[f]}\left(\frac{x}{y}\right) f(y,\tau), \tag{55}$$

where $\tau = \ln Q^2$ and $P^{[f]}$ are the splitting functions. Using moments $A_n^{[f]}$ of these splitting functions—the anomalous dimensions—this is

$$\frac{d}{d\tau} [f]_n(\tau) = \frac{\alpha_s(\tau)}{2\pi} A_n^{[f]} [f]_n(\tau). \tag{56}$$

This applies to the leading order results for the non-singlet twist-2 functions (with the usual + prescription) [31,32],

$$P^{[f_1]}(\beta) = P^{[g_1]}(\beta) = C_F \left[\frac{3}{2}\delta(1-\beta) + \frac{1+\beta^2}{(1-\beta)_+}\right], \tag{57}$$

$$P^{[h_1]}(\beta) = C_F \left[\frac{3}{2}\delta(1-\beta) + \frac{2\beta}{(1-\beta)_+}\right], \tag{58}$$

and the large N_c result in leading order for the interaction-dependent functions [33]

$$P^{[\tilde{f}]}(\beta) = \frac{N_c}{2} \left[\frac{1}{2}\delta(1-\beta) + \frac{2}{(1-\beta)_+} + c\right], \tag{59}$$

with $c = -1$ for $\tilde{g}_T$, $c = -3$ for $\tilde{h}_L$ and $c = +1$ for $\tilde{e}$. The corresponding anomalous dimensions are

$$A_n^{[f_1]} = A_n^{[g_1]} = C_F \left[\frac{3}{2} + \frac{1}{n(n+1)} - 2\sum_{j=1}^{n} \frac{1}{j}\right], \tag{60}$$

$$A_n^{[h_1]} = C_F \left[\frac{3}{2} - 2\sum_{j=1}^{n} \frac{1}{j}\right], \tag{61}$$

and for the interaction-dependent functions in the large N_c limit

$$A_n^{[\tilde{g}_T]} = \frac{N_c}{2} \left[\frac{1}{2} + \frac{1}{n} - 2\sum_{j=1}^{n} \frac{1}{j}\right], \tag{62}$$

$$A_n^{[\tilde{h}_L]} = \frac{N_c}{2} \left[\frac{1}{2} - \frac{1}{n} - 2\sum_{j=1}^{n} \frac{1}{j}\right], \tag{63}$$

$$A_n^{[\tilde{e}]} = \frac{N_c}{2} \left[\frac{1}{2} + \frac{3}{n} - 2\sum_{j=1}^{n} \frac{1}{j}\right]. \tag{64}$$

Using the moment analysis of the previous section, it is straightforward to find that the evolution of $g_{1T}^{(1)}$ is driven not only by this function itself but also by a higher moment of g_1 and a similar situation for $h_{1L}^{\perp(1)}$. In the large N_c limit ($C_F \to N_c/2$) one obtains

(omitting mass terms)

$$\frac{d}{d\tau}\big[g_{1T}^{(1)}\big]_n = \frac{\alpha_s(\tau)}{4\pi}\,N_c\left\{\left[\frac{1}{2} - \frac{1}{n+1} - 2\sum_{j=1}^{n}\frac{1}{j}\right]\big[g_{1T}^{(1)}\big]_n + \frac{1}{n+2}\,[g_1]_{n+1}\right\}, \quad (65)$$

$$\frac{d}{d\tau}\big[h_{1L}^{\perp(1)}\big]_n = \frac{\alpha_s(\tau)}{4\pi}\,N_c\left\{\left[\frac{1}{2} - \frac{3}{n+1} - 2\sum_{j=1}^{n}\frac{1}{j}\right]\big[h_{1L}^{\perp(1)}\big]_n - \frac{1}{n+1}\,[h_1]_{n+1}\right\}, \quad (66)$$

or in terms of the functions of light-cone momentum fractions

$$\frac{d}{d\tau}\,g_{1T}^{(1)}(x,\tau) = \frac{\alpha_s(\tau)}{4\pi}\,N_c\int_x^1 dy\left\{\left[\frac{1}{2}\,\delta(y-x) + \frac{x^2+xy}{y^2(y-x)_+}\right]g_{1T}^{(1)}(y,\tau) + \frac{x^2}{y^2}\,g_1(y,\tau)\right\}, \quad (67)$$

$$\frac{d}{d\tau}\,h_{1L}^{\perp(1)}(x,\tau) = \frac{\alpha_s(\tau)}{4\pi}\,N_c\int_x^1 dy\left\{\left[\frac{1}{2}\,\delta(y-x) + \frac{3x^2-xy}{y^2(y-x)_+}\right]h_{1L}^{\perp(1)}(y,\tau) - \frac{x}{y}\,h_1(y,\tau)\right\}. \quad (68)$$

Next we note that apart from a γ_5 matrix the operator structures of the T-odd functions $f_{1T}^{\perp(1)}$ and $h_1^{\perp(1)}$ are in fact the same as those of $g_{1T}^{(1)}$ and $h_{1L}^{\perp(1)}$ (they can be considered as the imaginary part of these functions [30]). This implies that for the non-singlet functions, one immediately can obtain the evolution of the T-odd functions,

$$\frac{d}{d\tau}\big[f_{1T}^{\perp(1)}\big]_n = \frac{\alpha_s(\tau)}{4\pi}\,N_c\left[\frac{1}{2} - \frac{1}{n+1} - 2\sum_{j=1}^{n}\frac{1}{j}\right]\big[f_{1T}^{\perp(1)}\big]_n, \quad (69)$$

$$\frac{d}{d\tau}\big[h_1^{\perp(1)}\big]_n = \frac{\alpha_s(\tau)}{4\pi}\,N_c\left[\frac{1}{2} - \frac{3}{n+1} - 2\sum_{j=1}^{n}\frac{1}{j}\right]\big[h_1^{\perp(1)}\big]_n. \quad (70)$$

Furthermore, for the chiral-odd functions, which do not mix with a gluon distribution, there is no difference between the non-singlet and the singlet evolution.

In the large N_c limit, the evolution equations for the non-singlet T-odd functions are of simple diagonal form with splitting functions

$$P^{[f_{1T}^{\perp(1)}]}(\beta) = \frac{N_c}{2}\left[\frac{1}{2}\,\delta(1-\beta) + \frac{\beta+\beta^2}{(1-\beta)_+}\right], \quad (71)$$

$$P^{[h_1^{\perp(1)}]}(\beta) = \frac{N_c}{2}\left[\frac{1}{2}\,\delta(1-\beta) + \frac{3\beta^2-\beta}{(1-\beta)_+}\right]. \quad (72)$$

Actually, we also obtain the anomalous dimensions (and splitting functions) of the T-odd twist-3 functions using $A^{[f_T]} = A^{[\tilde{f}_T]} = A^{[\tilde{g}_T]}$, $A^{[h]} = A^{[\tilde{h}]} = A^{[\tilde{h}_L]}$ and $A^{[e_L]} = A^{[\tilde{e}_L]} = A^{[e]} = A^{[\tilde{e}]}$.

6. Fragmentation functions

Just as for the distribution functions one can perform an analysis of the soft part describing quark fragmentation. One needs [5]

$$\Delta_{ij}(z,\boldsymbol{k}_T) = \sum_X \int \frac{d\xi^+ \, d^2\boldsymbol{\xi}_T}{(2\pi)^3} \, e^{ik\cdot\xi} \, \mathrm{Tr}\langle 0|\mathcal{U}(\infty,\xi)\psi_i(\xi)|P_h,X\rangle \times \langle P_h,X|\bar{\psi}_j(0)\mathcal{U}(0,\infty)|0\rangle \Big|_{\xi^-=0}. \tag{73}$$

Note that because of the definition of the light-like vectors $n_\pm$ (via P and P_h), the role of these vectors for fragmentation functions will be interchanged with respect to the distribution functions. For the production of unpolarized or spin-(1/2) hadrons h in semi-inclusive hard scattering processes one needs to leading order in $1/Q$ the correlation function [10]

$$\begin{aligned}\Delta(z,\boldsymbol{k}_T) = {} & z D_1\big(z,\boldsymbol{k}_T'^2\big)\not{n}_- + z D_{1T}^\perp\big(z,\boldsymbol{k}_T'^2\big)\frac{\epsilon_{\mu\nu\rho\sigma}\gamma^\mu n_-^\nu k_T^\rho S_{hT}^\sigma}{M_h} \\ & - z G_{1s}(z,\boldsymbol{k}_T')\not{n}_-\gamma_5 + z H_{1T}\big(z,\boldsymbol{k}_T'^2\big)\frac{[\not{S}_{hT},\not{n}_-]\gamma_5}{2} \\ & + z H_{1s}^\perp(z,\boldsymbol{k}_T')\frac{[\not{k}_T,\not{n}_-]\gamma_5}{2M_h} + i z H_1^\perp\big(z,\boldsymbol{k}_T'^2\big)\frac{[\not{k}_T,\not{n}_-]}{2M_h} + \mathcal{O}\left(\frac{M_h}{P_h^-}\right).\end{aligned} \tag{74}$$

We used the shorthand notation

$$G_{1s}(z,\boldsymbol{k}_T) \equiv S_{hL} G_{1L}\big(z,\boldsymbol{k}_T'^2\big) + \frac{(\boldsymbol{k_T}\cdot\boldsymbol{S}_{hT})}{M_h} G_{1T}\big(z,\boldsymbol{k}_T'^2\big), \tag{75}$$

etc. The arguments of the fragmentation functions are $z = P_h^-/k^-$ and $\boldsymbol{k}_T' = -z\boldsymbol{k}_T$. The first is the (light-cone) momentum fraction of the produced hadron, the second is the transverse momentum of the produced hadron with respect to the quark. The $\boldsymbol{k}_T$-integrated results are, using $F(z) \equiv \int d^2k_T' \, F(z,\boldsymbol{k}_T'^2)$ and $F^{(n)}(z) \equiv \int d^2k_T' \, (\boldsymbol{k}_T^2/2M_h^2)^n F(z,\boldsymbol{k}_T'^2) = \int d^2k_T' \, (\boldsymbol{k}_T'^2/2z^2M_h^2)^n F(z,\boldsymbol{k}_T'^2)$,

$$\Delta^{\text{twist-2}}(z) = \frac{D_1(z)}{z}\not{n}_- + S_{hL}\frac{G_1(z)}{z}\gamma_5\not{n}_- + \frac{H_1(z)}{z}\gamma_5\not{S}_{hT}\not{n}_-, \tag{76}$$

$$\begin{aligned}\Delta^{\text{twist-3}}(x) = {} & \frac{M_h}{P_h^-}\left\{\frac{E(z)}{z} + \frac{G_T(z)}{z}\gamma_5\not{S}_{hT} + S_{hL}\frac{H_L(z)}{z}\gamma_5\frac{[\not{n}_-,\not{n}_+]}{2}\right\} \\ & + \frac{M_h}{P_h^-}\left\{-iS_{hL}\frac{E_L(z)}{z}\gamma_5 - \frac{D_T(z)}{z}\epsilon_T^{\rho\sigma}\gamma_\rho S_{hT\sigma} + i\frac{H(z)}{z}\frac{[\not{n}_-,\not{n}_+]}{2}\right\},\end{aligned} \tag{77}$$

$$\begin{aligned}\Delta_\partial^\alpha(z) = {} & -\frac{G_{1T}^{(1)}(z)}{z}S_{hT}^\alpha\not{n}_-\gamma_5 - S_{hL}\frac{H_{1L}^{\perp(1)}(z)}{z}\frac{[\gamma^\alpha,\not{n}_-]\gamma_5}{2} \\ & - \frac{D_{1T}^{\perp(1)}(z)}{z}\epsilon^\alpha{}_{\mu\nu\rho}\gamma^\mu n_+^\nu S_{hT}^\rho - i\frac{H_1^{\perp(1)}(z)}{z}\frac{[\gamma^\alpha,\not{n}_-]}{2}.\end{aligned} \tag{78}$$

In the twist-3 functions one can again isolate the interaction-dependent parts as done for the distribution functions. They are now given by

$$\widetilde{G}_T(z) = G_T(z) - zG_{1T}^{(1)}(z) - \frac{m}{M_h}\, zH_1(z), \tag{79}$$

$$\widetilde{H}_L(z) = H_L(z) + 2zH_{1L}^{\perp(1)}(z) - \frac{m}{M_h}\, zG_1(z), \tag{80}$$

$$\widetilde{E}(z) = E(z) - \frac{m}{M_h}\, zD_1(z), \tag{81}$$

$$\widetilde{D}_T(z) = D_T(z) + zD_{1T}^{\perp(1)}(z), \tag{82}$$

$$\widetilde{H}(z) = H(z) + 2zH_1^{\perp(1)}(z), \tag{83}$$

$$\widetilde{E}_L(z) = E_L(z). \tag{84}$$

For the $\boldsymbol{k}_T$-integrated or the $\boldsymbol{k}_T^2/2M_h$-weighted fragmentation functions all results are obtained from the distribution functions by replacing $x \to 1/z$ and $f_{\dots}(x) \to D_{\dots}(z)/z$, $g_{\dots}(x) \to G_{\dots}(z)/z$ and $h_{\dots}(x) \to H_{\dots}(z)/z$. The same applies to the relations from Lorentz invariance [10,34]

$$G_T(z) = G_1(z) - z^3\, \frac{d}{dz}\left[\frac{G_{1T}^{(1)}(z)}{z}\right], \tag{85}$$

$$H_L(z) = H_1(z) + z^3\, \frac{d}{dz}\left[\frac{H_{1L}^{\perp(1)}(z)}{z}\right], \tag{86}$$

$$D_T(z) = z^3\, \frac{d}{dz}\left[\frac{D_{1T}^{\perp(1)}(z)}{z}\right], \tag{87}$$

$$H(z) = z^3\, \frac{d}{dz}\left[\frac{H_1^{\perp(1)}(z)}{z}\right]. \tag{88}$$

Expressing the functions in twist-2 functions and interaction-dependent functions gives

$$\begin{aligned}\frac{G_T(z)}{z} = &-\int_z^1 dy\, \frac{G_1(y)}{y^2} + \frac{m}{M_h}\left[H_1(z) + \int_z^1 dy\, \frac{H_1(y)}{y}\right] \\ &+ \left[\frac{\widetilde{G}_T(z)}{z} + \int_z^1 dy\, \frac{\widetilde{G}_T(y)}{y^2}\right],\end{aligned} \tag{89}$$

$$G_{1T}^{(1)}(z) = -\int_z^1 dy\, \frac{G_1(y)}{y^2} + \frac{m}{M_h}\int_z^1 dy\, \frac{H_1(y)}{y} + \int_z^1 dy\, \frac{\widetilde{G}_T(y)}{y^2}, \tag{90}$$

$$\begin{aligned}H_L(z) = &-2\int_z^1 dy\, \frac{H_1(y)}{y} + \frac{m}{M_h}\left[zG_1(z) + 2\int_z^1 dy\, G_1(y)\right] \\ &+ \left[\widetilde{H}_L(z) + 2\int_z^1 dy\, \frac{\widetilde{H}_L(y)}{y}\right],\end{aligned} \tag{91}$$

$$zH_{1L}^{\perp(1)}(z) = \int_z^1 dy\,\frac{H_1(y)}{y} - \frac{m}{M_h}\int_z^1 dy\,G_1(y) - \int_z^1 dy\,\frac{\widetilde{H}_L(y)}{y}, \tag{92}$$

$$E(z) = \widetilde{E}(z) + \frac{m}{M_h}\,zD_1(z), \tag{93}$$

$$\frac{D_T(z)}{z} = \left[\frac{\widetilde{D}_T(z)}{z} + \int_z^1 dy\,\frac{\widetilde{D}_T(y)}{y^2}\right], \tag{94}$$

$$D_{1T}^{\perp(1)}(z) = -\int_z^1 dy\,\frac{\widetilde{D}_T(y)}{y^2}, \tag{95}$$

$$H(z) = \left[\widetilde{H}(z) + 2\int_z^1 dy\,\frac{\widetilde{H}(y)}{y}\right], \tag{96}$$

$$zH_1^{\perp(1)}(z) = -\int_z^1 dy\,\frac{\widetilde{H}(y)}{y}, \tag{97}$$

$$E_L(z) = \widetilde{E}_L(z). \tag{98}$$

The relations for the moments of fragmentation functions can be obtained from the above equations or from the results of the distribution functions via the replacements $n \to -n$ followed by $[f]_{-n} \to [D/z]_n = [D]_{n-1}$. This yields

$$[G_T]_n = -\frac{1}{n+1}[G_1]_n + \frac{n+2}{n+1}\big[\widetilde{G}_T\big]_n + \frac{m}{M_h}\frac{n+2}{n+1}[H_1]_{n+1}, \tag{99}$$

$$\big[G_{1T}^{(1)}\big]_{n+1} = -\frac{1}{n+1}\left([G_1]_n - \big[\widetilde{G}_T\big]_n - \frac{m}{M_h}[H_1]_{n+1}\right), \tag{100}$$

$$[H_L]_n = -\frac{2}{n}[H_1]_n + \frac{n+2}{n}\big[\widetilde{H}_L\big]_n + \frac{m}{M_h}\frac{n+2}{n}[G_1]_{n+1}, \tag{101}$$

$$\big[H_{1L}^{\perp(1)}\big]_{n+1} = \frac{1}{n}\left([H_1]_n - \big[\widetilde{H}_L\big]_n - \frac{m}{M_h}[G_1]_{n+1}\right), \tag{102}$$

$$[E]_n = \big[\widetilde{E}\big]_n + \frac{m}{M_h}[D_1]_{n+1}, \tag{103}$$

$$[D_T]_n = \frac{n+2}{n+1}\big[\widetilde{D}_T\big]_n, \tag{104}$$

$$\big[D_{1T}^{\perp(1)}\big]_{n+1} = -\frac{1}{n+1}\big[\widetilde{D}_T\big]_n, \tag{105}$$

$$[H]_n = \frac{n+2}{n}\big[\widetilde{H}\big]_n, \tag{106}$$

$$\big[H_1^{\perp(1)}\big]_{n+1} = -\frac{1}{n}\big[\widetilde{H}\big]_n, \tag{107}$$

$$[E_L]_n = \big[\widetilde{E}_L\big]_n. \tag{108}$$

The autonomous evolution equations are again of the form

$$\frac{d}{d\tau} D(z,\tau) = \frac{\alpha_s(\tau)}{2\pi} \int_z^1 \frac{dy}{y}\, P^{[D]}\left(\frac{z}{y}\right) D(y,\tau), \tag{109}$$

or via the (usual) moments $A_n^{[D]} = \int_0^1 dz\, z^{n-1} P^{[D]}(z)$ of the splitting functions,

$$\frac{d}{d\tau} [D]_n(\tau) = \frac{\alpha_s(\tau)}{2\pi} A_n^{[D]} [D]_n(\tau). \tag{110}$$

For the leading order contributions the analytic structure of the corrections for fragmentation functions is similar as for distribution functions. We note a (generalized) Gribov–Lipatov reciprocity, summarized by the following procedure. The splitting functions for distribution functions $f(x,\tau)$ and corresponding fragmentation functions $zD(z,\tau)$ are related by

$$P^{[f]}(\beta) = \frac{\mathcal{N}(\beta)}{(1-\beta)_+}, \tag{111}$$

$$P^{[zD]}(\beta) = \frac{\beta^2 \mathcal{N}(1/\beta)}{(1-\beta)_+}. \tag{112}$$

This relation works for the twist-2 fragmentation functions *and* the interaction-dependent functions [28], for $\mathcal{N}(\beta)$ being (at most a quadratic) polynomial in β. In the case of the twist-2 functions the functional form of the splitting functions is the same for distribution and fragmentation functions. This is no longer true for the interaction-dependent functions. For the anomalous dimensions of distribution and fragmentation functions the relation becomes

$$A_n^{[f]} = \mathcal{A}(n) - 2\sum_{j=1}^{n} \frac{1}{j} = \mathcal{A}(n) - 2\gamma_E - 2\psi(n+1), \tag{113}$$

$$A_{n+1}^{[D]} = \mathcal{A}\big(-(n+1)\big) - 2\gamma_E - 2\psi(n+1) = \mathcal{A}\big(-(n+1)\big) - 2\sum_{j=1}^{n} \frac{1}{j}, \tag{114}$$

where $\mathcal{A}(n)$ is a rational function. We have not yet investigated the wider applicability of the above relations. We find for the twist-2 fragmentation functions the familiar results, which obey the original Gribov–Lipatov reciprocity relation $A_n^{[f]} = A_{n+1}^{[D]}$ between the twist-2 distribution functions $f = f_1, g_1, h_1$ and fragmentation functions $D = D_1, G_1, H_1$,

$$A_{n+1}^{[D_1]} = A_{n+1}^{[G_1]} = C_F \left[\frac{3}{2} + \frac{1}{n(n+1)} - 2\sum_{j=1}^{n} \frac{1}{j}\right], \tag{115}$$

$$A_{n+1}^{[H_1]} = C_F \left[\frac{3}{2} - 2\sum_{j=1}^{n} \frac{1}{j}\right]. \tag{116}$$

In the large N_c limit, our generalized reciprocity relations in Eqs. (113) and (114) applied to Eqs. (62)–(64) give the results for the interaction-dependent functions [28,29]

$$A_{n+1}^{[\widetilde{G}_T]} = \frac{N_c}{2}\left[\frac{1}{2} - \frac{1}{n+1} - 2\sum_{j=1}^{n}\frac{1}{j}\right], \tag{117}$$

$$A_{n+1}^{[\widetilde{H}_L]} = \frac{N_c}{2}\left[\frac{1}{2} + \frac{1}{n+1} - 2\sum_{j=1}^{n}\frac{1}{j}\right], \tag{118}$$

$$A_{n+1}^{[\widetilde{E}]} = \frac{N_c}{2}\left[\frac{1}{2} - \frac{3}{n+1} - 2\sum_{j=1}^{n}\frac{1}{j}\right]. \tag{119}$$

Again one then also knows $A^{[D_T]} = A^{[\widetilde{D}_T]} = A^{[\widetilde{G}_T]}$, $A^{[H]} = A^{[\widetilde{H}]} = A^{[\widetilde{H}_L]}$ and $A^{[E_L]} = A^{[\widetilde{E}_L]} = A^{[E]} = A^{[\widetilde{E}]}$.

Using the moment analysis (the reciprocity relations cannot be used straightforwardly) one obtains, omitting the mass terms,

$$\begin{aligned}&\frac{d}{d\tau}\left[G_{1T}^{(1)}\right]_{n+1}\\ &\quad = \frac{\alpha_s(\tau)}{4\pi}N_c\left\{\left[\frac{1}{2} + \frac{1}{n} - 2\sum_{j=1}^{n}\frac{1}{j}\right]\left[G_{1T}^{(1)}\right]_{n+1} - \frac{n}{(n-1)(n+1)}\left[G_1\right]_n\right\},\end{aligned} \tag{120}$$

$$\frac{d}{d\tau}\left[H_{1L}^{\perp(1)}\right]_{n+1} = \frac{\alpha_s(\tau)}{4\pi}N_c\left\{\left[\frac{1}{2} + \frac{3}{n} - 2\sum_{j=1}^{n}\frac{1}{j}\right]\left[H_{1L}^{\perp(1)}\right]_{n+1} + \frac{n-1}{n^2}\left[H_1\right]_n\right\}, \tag{121}$$

with in this case mixing with a lower moment of the twist-2 functions. In terms of the functions of light-cone momentum fractions one finds

$$\begin{aligned}\frac{d}{d\tau}zG_{1T}^{(1)}(z,\tau) = \frac{\alpha_s(\tau)}{4\pi}N_c\int_z^1 dy\,\Bigg\{&\left[\frac{1}{2}\delta(y-z) + \frac{y+z}{y(y-z)_+}\right]yG_{1T}^{(1)}(y,\tau)\\ &- \frac{y^2+z^2}{2y^2z}G_1(y,\tau)\Bigg\},\end{aligned} \tag{122}$$

$$\begin{aligned}\frac{d}{d\tau}zH_{1L}^{\perp(1)}(z,\tau) = \frac{\alpha_s(\tau)}{4\pi}N_c\int_z^1 dy\,\Bigg\{&\left[\frac{1}{2}\delta(y-z) + \frac{3y-z}{y(y-z)_+}\right]yH_{1L}^{\perp(1)}(y,\tau)\\ &+ \frac{1+\ln(z/y)}{y}H_1(y,\tau)\Bigg\}.\end{aligned} \tag{123}$$

Given the fact that, apart from an additional γ_5, the operator structure for the T-odd Sivers and Collins functions, $D_{1T}^{\perp(1)}$ and $H_1^{\perp(1)}$, are the same as those of $G_{1T}^{(1)}$ and $H_{1L}^{\perp(1)}$ but without mixing with G_1 or H_1, one finds in the large N_c limit an autonomous evolution for the T-odd functions, with anomalous dimensions

$$A_{n+1}^{[D_{1T}^{\perp(1)}]} = \frac{N_c}{2}\left[\frac{1}{2} + \frac{1}{n} - 2\sum_{j=1}^{n}\frac{1}{j}\right], \tag{124}$$

$$A_{n+1}^{[H_1^{\perp(1)}]} = \frac{N_c}{2}\left[\frac{1}{2} + \frac{3}{n} - 2\sum_{j=1}^{n}\frac{1}{j}\right], \tag{125}$$

corresponding to splitting functions

$$P^{[zD_{1T}^{\perp(1)}]}(\beta) = \frac{N_c}{2}\left[\frac{1}{2}\delta(1-\beta) + \frac{1+\beta}{(1-\beta)_+}\right], \tag{126}$$

$$P^{[zH_1^{\perp(1)}]}(\beta) = \frac{N_c}{2}\left[\frac{1}{2}\delta(1-\beta) + \frac{3-\beta}{(1-\beta)_+}\right]. \tag{127}$$

The results Eqs. (125) and (127) should prove useful for studies of the Collins effect and Eqs. (124) and (126) for studies of transversely polarized Λ production [35].

7. Discussion and conclusions

Our goal was to obtain the evolution equations of the functions that appear in azimuthal spin asymmetries. These p_T-dependent functions appear in asymmetries that are not suppressed by explicit powers of the hard momentum. But as functions of transverse momentum they are not of definite twist, which implies that in order to obtain the evolution equations one has to calculate corrections to higher twist operators as well. For the first $\boldsymbol{p}_T^2/2M^2$ moment (transverse moment) of these p_T-dependent functions, such as for the Collins fragmentation function $H_1^{\perp(1)} = \int d^2k_T'\, \boldsymbol{k}_T'^2/2z^2M_h^2 H_1^{\perp}(z, \boldsymbol{k}_T'^2)$, we obtain DGLAP-like evolution equations. Such moments appear in cross sections weighted with the momentum q_T^α, where only the directional (azimuthal) dependence remains. For explicit examples we refer to Refs. [11,13]. In case one does not weight the transverse momentum integration of the differential cross section, one is only sensitive to the twist-2 functions f_1, g_1 and h_1 (and their fragmentation counterparts), but in case one weighs with one or more powers of the observed transverse momentum, one becomes sensitive to the functions $g_{1T}^{(1)}$, $h_{1L}^{\perp(1)}$, $f_{1T}^{\perp(1)}$, $h_1^{\perp(1)}$ (and their fragmentation counterparts), which are functions of the light-cone momentum fraction x only.

In the large-N_c limit, the non-singlet evolution of these functions involves only the functions themselves and (in the T-even case) only well-known twist-2 functions. For the chiral-odd functions the equations also apply to the singlet case, since there is no mixing with gluon distribution functions. The large-N_c evolution equations are expected to be good approximations to the full evolution equations which are not of this simple form (cf. Ref. [36]), because of the appearance of two-argument twist-3 functions as in Eq. (10). It is not excluded that the first $1/N_c$ correction to the result obtained here may still lead to autonomous evolution equations, but we will not address this issue here. Especially the (large N_c) evolution equation we have obtained for $H_1^{\perp(1)}$,

$$\frac{d}{d\tau} zH_1^{\perp(1)}(z,\tau) = \frac{\alpha_s}{4\pi} N_c \int_z^1 dy \left[\frac{1}{2}\delta(y-z) + \frac{3y-z}{y(y-z)_+}\right] yH_1^{\perp(1)}(y,\tau), \tag{128}$$

should prove useful for the comparison of data on Collins function asymmetries from different experiments, performed at different energies.

It is worth investigating the large Q behavior of the solutions to the various evolution equations. For this purpose we have given the first 3 anomalous dimensions for the different

Table 1
The anomalous dimensions from which the large Q^2 behavior of the moments, proportional to $[\alpha_s(Q^2)]^{d_n}$, is obtained. Defining the moments a_n taking out the factor C_F or $N_c/2$ from the anomalous dimensions A_n, one has for the twist-2 functions $d_n = -2a_n C_F/b_0$ with $b_0 = (11N_c - 2N_f)/3$, while for the large N_c results one has $d_n = -3a_n/11$. Also indicated is the charge conjugation behavior of the functions, $\bar{f}(x) = \pm f(-x)$

Function	C	a_1	a_2	a_3	Validity
f_1	$-$	0	$-7/6$	$-25/12$	
g_1	$+$	0	$-7/6$	$-25/12$	
h_1	$-$	$-1/2$	$-3/2$	$-13/6$	
$\tilde{g}_T$ and $\tilde{f}_T$	$+$	$-1/2$	-2	$-17/6$	large N_c
$\tilde{h}_L$ and $\tilde{h}$	$-$	$-5/2$	-3	$-7/2$	large N_c
$\tilde{e}$	$+$	$+3/2$	-1	$-13/6$	large N_c
$g_{1T}^{(1)}$ and $f_{1T}^{\perp(1)}$	$-$	-2	$-17/6$	$-41/12$	large N_c
$h_{1L}^{\perp(1)}$ and $h_1^{\perp(1)}$	$+$	-3	$-7/2$	$-47/12$	large N_c
zD_1	$-$	0	$-7/6$	$-25/12$	
zG_1	$+$	0	$-7/6$	$-25/12$	
zH_1	$-$	$-1/2$	$-3/2$	$-13/6$	
$z\widetilde{G}_T$ and $z\widetilde{D}_T$	$+$	-2	$-17/6$	$-41/12$	large N_c
$z\widetilde{H}_L$ and $z\widetilde{H}$	$-$	-1	$-13/6$	$-35/12$	large N_c
$z\widetilde{E}$	$+$	-3	$-7/2$	$-47/12$	large N_c
$zG_{1T}^{(1)}$ and $zD_{1T}^{\perp(1)}$	$-$	$-1/2$	-2	$-17/6$	large N_c
$zH_{1L}^{\perp(1)}$ and $zH_1^{\perp(1)}$	$+$	$+3/2$	-1	$-13/6$	large N_c

functions in Table 1. First we note that all (diagonal) anomalous dimensions of $g_{1T}^{(1)}$, $h_{1L}^{\perp(1)}$, $f_{1T}^{\perp(1)}$ and $h_1^{\perp(1)}$ are negative, implying that these functions will vanish asymptotically ($Q^2 \to \infty$), except that for the T-even functions there is mixing with g_1 and h_1, but this does not alter the conclusion.

For the fragmentation counterparts the conclusion is similar, except for the fact that the lowest anomalous dimensions of $zH_{1L}^{\perp(1)}$ and $zH_1^{\perp(1)}$ are positive, potentially leading to divergent behavior of the functions as $Q^2 \to \infty$. However, here we recall the Schäfer–Teryaev sum rule [37]

$$\int dz\, z H_1^{\perp(1)}(z) = 0, \tag{129}$$

which states that the first moment of $zH_1^{\perp(1)}$ is zero, making the sign of the first anomalous dimension irrelevant. Similar sum rules hold for the other first transverse moments of fragmentation functions [35,37]. All higher moments will vanish asymptotically. The behavior of the sum rule for the first moment of the function e is discussed in Ref. [3].

In conclusion, using Lorentz invariance and the QCD equations of motion, the operator structure of the transverse moments of p_T-dependent quark distribution and fragmentation functions can be found in terms of twist-2 and twist-3 operators. Knowing their, for large N_c simple, evolution one also knows the evolution of azimuthal asymmetries in semi-inclusive hard scattering processes.

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